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Neuroscience and Mathematics
From Inborn Skills to Cantor's Paradise[1]

Introduction: The Mathematical Subject

When one contemplates the history of the philosophy of mathematics, three kinds of mathematical subjects – i.e. subjects capable of carrying out mathematical reasoning – may be identified: the Platonic, the transcendental and the empirical.[2] The Platonic subject has the ability to 'see' mathematical objects, which belong to an independent, eternal reality. A 'strong' Platonic subject (God?) is capable of perceiving all the mathematical world at once, and so is capable of contemplating actual infinity; the 'weak' Platonic subject has some cognitive access to the mathematical universe – through some special faculty of mind such as intuition – but not all-encompassing.

The second kind of mathematical subject is the transcendental, a conception advocated vividly by Immanuel Kant and his followers (who include the most important philosophers of mathematics,

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[2] See A. Olszewski, *Teza Churcha. Kontekst historyczno-filozoficzny*, Universitas, Kraków 2009, chapter 6.2, *passim*.

such as Brouwer, and – with some reservations – even Hilbert). The transcendental subject *constructs* mathematical objects, and so – in a way – they are not independent of it. However, the use of the word 'construct' may be misleading here. The transcendental subject is not a physically existing – nor physically realizable – entity. It is an ideal projection of human mathematical capacities. In other words, the transcendental subject stands *vis a vis* the entire body of possible mathematical knowledge; it is a postulate of what *can be done* in mathematics, disregarding any physical limitations such as time or space. Thus, when we speak of 'constructions of mathematical objects', we are referring to what is constructable in principle, not actually. To put it in a different way: human mathematical practice not only can never transgress the boundaries set by the ideal of the transcendental subject, but it also can never reach them. Because of its characteristics, the transcendental subject cannot perceive (construct) actual infinity, but is capable of grasping potential infinity.

Finally, there exists the notion of an empirical mathematical subject, i.e. someone whose mathematical capacities are limited by spatio-temporal boundaries. For instance, a universal Turing machine is *not* a model of an empirical subject, as it utilizes an infinite tape and is capable of repeating its simple operations a number of times which is beyond the reach of any spatio-temporally limited agent.

It is my claim that – despite some attempts of the representatives of psychologism and similar stances – most of the philosophies of mathematics developed during the previous 200 years presuppose a Platonic or transcendental view of the mathematical subject. This is, naturally, an oversimplification, but since the rejection of psychologism at the turn of the 20th century, the abilities of mathematical subjects (e.g., Brouwer's creative subject, or the subject capable of manipulating symbols within Hilbert's formal systems) have always been understood in a non-empirical way. The spectacular advances in the neuroscientific studies of mathematical skills, achieved during the last two decades require, however, a re-thinking of the problem of the mathematical subject. So: is the empirical subject back?

In what follows I will sketch a conception of mathematics that is suggested by recent scientific findings. I will also identify some challenges to this view, and argue for a Popperian ontology of mathematics. I will conclude with an answer to the question posed at the end of the previous paragraph.

1. The Number Sense or the 'Embrained' Mathematics

During the last twenty years or so, a number of neuroscientific studies have been devoted to uncovering the origins of human mathematical capacities. The experiments in question include preferential looking, habituation of looking time, anticipatory head turning, explanatory reaching, neuroimaging with EEG or fMRI, but also the careful study of the mathematical skills of various animals (birds, non-human primates).[3]

To illustrate: an experiment of Izard *et al*. had the following setting: newborns were presented with a series of syllable trains of various pitch and duration. Half of the participants were exposed to 4-syllable trains, the other half – to 12-syllable trains. After a two minute time interval the infants were shown screens with visual arrays consisting of 4 or 12 objects. Their times of looking at the screen were recorded. It proved that they tended to look longer at the visual arrays that correspond to an ongoing sound sequence.[4]

Another example are the classical experiments of Starkey and Cooper. Infants were shown consecutive slides with two dots, but differing in size and the distance between them. The experimenters measured fixation time of the infants, i.e. the time they spent looking at a new slide. After a while the fixation times decreased – the new slides with two dots were looked at for shorter periods. Then, without

[3] Cf. E.S. Spelke, *Natural Number and Natural Geometry*, [in:] *Space, Time and Number in the Brain*, eds. S. Dehaene, E. Brannon, Academic Press, London 2011, pp. 287–317.

[4] *Ibid.*, p. 310.

a warning, slides with three dots appeared, and the infants exhibited longer fixation times, which suggested that they could perceive the difference between two and three.[5]

Yet another example of studying infants' mathematical abilities consisted in placing two puppets on a stage, covering them with a screen, and then visibly removing one of the puppets. When the screen was lowered and there was only one puppet, infants expressed no surprise; however, if there were still two puppets, the fixation time was longer, suggesting that the infants were surprised by what they saw.[6]

These and other results suggest that there exists an inborn capacity to 'deal' with small numbers; that this capacity is cross-modal (i.e., it 'extracts' numerosity independent of the mode of presentation – infants 'know' that three dots and three spoken syllables share the same numerosity); and that it enables understanding of simple mathematical operations performed on small numbers (addition or subtraction).

In fact, the current research implies that two separate brain systems are responsible for these capacities: the object tracking system (OTS) and the approximate number system (ANS). OTS is a system that enables tracking multiple individuals (up to 3 or 4). It is based on the principles of cohesion (moving objects are recognized as bounded wholes), continuity (objects move on unobstructed paths) and contact (objects do not interact at a distance).[7] The existence of the OTS system is confirmed by a number of tests, including visual short-term memory tasks, multiple-objects tracking tasks, or enumeration tasks. The last kind of tests confirms human ability of *subitizing*, i.e. of an instant and highly accurate determination of a number of object in small collections (3–4), even presented very briefly.[8] Furthermore,

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[5] Cf. G. Lakoff, R. Núñez, *Where Mathematics Comes From*, Basic Books, New York 2000, p. 16.

[6] *Ibid*.

[7] M. Piazza, *Neurocognitive Start-Up Tools for Symbolic Number Representations*, [in:] *Space, Time and Number…*, *op.cit*., p. 270.

[8] *Ibid*., p. 271.

it is speculated that the posterior parietal and occipital regions of the brain play the crucial role in the performance of such task, which suggests that these regions are the location of OTS.[9]

ANS, on the other hand, is a system for representing the approximate number of items in sets. It works according to the famous Weber's Law: the threshold of discrimination between two stimuli increases linearly with stimulus intensity. The law postulates a logarithmic relation between the physical stimulus and its internal representation. In the case of ANS, the Weber fraction, or the smallest variation to a quantity that can be readily perceived, changes over human development. For newborns it is 1:3, for 6-month-old babies it is 1:2, for 1-year-old children it is 2:3, for a 4-year-old it is 3:4, for a 7-year-old it is 4:5, while for a 20-year-old it is 7:8. It means that a newborn can discriminate between 1 and 3, or 2 and 6, or 10 and 30, but not 1 and 2, 2 and 5, or 10 and 27. Four-year-old children can tell that there is a difference in numerosity between sets consisting of 6 and 8 or 12 and 16 elements, but not 7 and 8 or 12 and 15. An adults' ANS system is even more 'sensitive': they can discriminate (without counting) between sets consisting of 14 and 16 elements or 70 and 80 elements, but not 70 and 78 elements. It is quite well established that ANS is located in the mid-intreparietal sulcus.[10]

One important difference between OTS and ANS must be noted here. ANS may be considered a core system *dedicated to numbers*. Its only task is to assess the approximate numerosities of sets. OTS, on the other hand, serves object tracking, and 'takes notice' of various aspects of the tracked objects such as their shape, colour, etc.

The existence of basic mathematical skills was also found in animals. Interestingly enough, their numerical skills are ratio-dependent, suggesting that they are equipped with a predecessor of the human ANS system. For instance, it has been demonstrated that "macaque monkeys spontaneously match the approximate number of individuals

[9] *Ibid.*, p. 270.
[10] *Ibid.*, pp. 268–269.

they see to the number of individual voices they hear; and also sum up visual and auditory stimuli to estimate their total number, without previous training."[11] This suggests that ANS is an old evolutionary device.

There are also other basic problems connected to mathematical thinking that generate much interest among neuroscientists. One such example is the problem of how the representation of numbers and complex counting is possible, given that working memory can store up to only 4 visual items. Feigenson speculates that this is possible thanks to the flexibility of the working memory – various things can count as items: individual objects (representations of exact numbers of objects), sets of objects or ensembles (representations of an approximate number of objects).[12] Another problem is the question of whether the process of counting depends on eye movement, as some suggest. Cavanagh and He claim that eye movement is not essential to counting; what is required is a set of attention pointers that individuate targets of interest in the visual field.[13]

Much attention is also devoted to the relationship between the representations of space, time and number. The research in this area was stimulated by the discovery of the SNARC effect by Dehaene and his team in 1993.[14] The experiment setting was as follows: adults were to decide whether the displayed number (in Arabic notation) is odd or even. In the former case, they were to press the left button, in the latter – the right one. The interesting observation was that large numbers led to faster key presses on the right-hand side of the space, irrespective of whether they were odd or even. This suggested that there is some

[11] *Ibid.*, p. 269.

[12] Cf. L. Feigenson, *Objects, Sets, and Ensembles*, [in:] *Space, Time and Number…*, *op.cit.*, pp. 13–22.

[13] Cf. P. Cavanagh, S. He, *Attention Mechanisms for Counting in Stabilized and in Dynamic Displays*, [in:] *Space, Time and Number…*, *op.cit.*, pp. 23–25.

[14] Cf. S. Dehaene *et al.*, *The Mental Representation of Parity and Number Magnitude*, "Journal of Experimental Psychology: General" 1993, vol. 122, pp. 371–396; W. Fias, J.-Ph. van Dijck, W. Gevers, *How Is Number Associated with Space? The Role of Working Memory*, [in:] *Space, Time and Number…*, *op.cit.*, pp. 133–148.

relationship between number and space representations. In particular, it may be speculated that "mental representation of numbers takes the form of a horizontally oriented line which is functionally homeomorphic to the way physical lines are represented"[15] (the so-called number line hypothesis). This would explain why large numbers lead to the preference of the right-hand side (on the number line, the larger the number is, the more to the right it is located). Other SNARC effects have also been discovered (associating numbers not only with space, but also with temporal duration, objects size or finger location) leading, e.g., to Vincent Walsh's ATOM theory (a theory of magnitude), which postulates a single magnitude mechanism that underlies the representations of number, space and time. Other researchers disagree with this proposal, suggesting other solutions, such as partially shared accumulator systems (Deheane), or the implication of the role of the serial position in working memory as an important determinant of the interactions between number and space (Fias).[16]

Altogether, the current research in the neurobiological foundations of mathematical knowledge clearly suggest that there exists an inborn – or, to coin a word, an *embrained* – set of very basic mathematical skills, which – in the context of arithmetic capabilities – consists of OTS and ANS. It must also be noted that similar inborn basic skills are implicated in the case of geometrical reasoning (the 2D and 3D systems).[17] For the sake of readability, I limit myself to the arithmetic capacities, but the conclusions I draw below are applicable, *mutatis mutandis*, to geometry.

15 W. Fias, J.-Ph. van Dijck, W. Gevers, *How is Number Associated with Space? The Role…*, *op.cit.*, p. 133.

16 See *ibid.*, pp. 139–144.

17 Cf. E.S. Spelke, *Natural Number…*, *op.cit.*

2. The Bumpy Road to Cantor's Paradise or Embodied Mathematics

The inborn or 'embrained' arithmetic is quite limited. The question thus is, how to get from the OTS and ANS systems to the beautiful but complex world of contemporary mathematics, with its differential equations, Noether rings, Hilbert spaces, non-commutative geometries, quaternions, and infinities. It is no surprise that – at least for now – neuroscience has little to say about this problem, and what it says is quite speculative.

Even the simplest of issues is controversial: what is the mechanism that enables us to break the 'number 4' barrier and acquire the competence to use natural numbers? Developmental psychology tells us that this process is very slow. At 6 months, children can add and subtract one; at two they begin to learn sequences of counting words, but do not map the words onto the numbers they represent; half a year later they recognize that number words mean more than one; at four, children can use fingers to aid adding; at five and a half they understand the commutativity of addition; a year later they understand the complementarity of addition and subtraction. More importantly, between 2. and 4. year, children learn to map number words '1', '2', '3' and '4' to the corresponding cardinalities one after another, but it may take up to six months to move to the next number.[18]

The first hypothesis of how an individual moves from counting from 1–4 to greater numbers has been developed by Carey. She calls the process bootstrapping. It consists in realizing that – when a children hears 'five' – the approximate numerosity of 'fiveness' is activated (in ANS) and then – on the basis of what the child knows about numbers 1–4 (from the workings of the 'exact', OTS system) – she concludes that 5 is also an exact number. Carey believes, then, that the essential move from '4' to '5' results from the interplay between both core mathematical systems: OTS and ANS. A child learns

[18] Cf. *ibid*.

what are exact numbers thanks to the way OTS operates; then, on the basis of the approximate representation of ‘fiveness’, a kind of inductive step is made, which enables the move to greater natural numbers.[19]

A different proposal has been developed by Piazza.[20] She observes that the ANS system may be used to represent not only large numbers, but also small ones. Moreover, ANS quite quickly becomes very precise as regards small numerosities. Given the progression in the sensitivity of ANS (the changes to the Weberian ratio), in order to distinguish between 2, 3, and larger numbers a ratio of 3:4 is needed. This happens at around three years of age, and coincides with the period when children become ‘three-knowers’. In other words, Piazza believes that no interplay between OTS and ANS is needed to ‘break the number four threshold’ – the increasing precision of the ANS system is sufficient to account for this ability. She supports this hypothesis with the following observations. Firstly, it is true that before children learn how to count, their arithmetic abilities are limited to the initial four natural numbers; on the other hand, however, there is no evidence suggesting that OTS plays any role after counting skills are acquired. It is thus difficult to understand how a system that serves no purpose in full-blooded counting would be so essential in acquiring the counting skills. Secondly, the fact that it takes some time for a child to get from understanding that the word ‘one’ corresponds to the number 1, to understanding that the word ‘two’ corresponds to the number two, and then again from 2 to 3, and from 3 to 4, may serve as an argument against the hypothesis that OTS plays some significant role in the development of numerical skills – if it did, it would be natural to assume that children learn the association of number words with the four corresponding numbers at the same age. Thirdly, there is no data suggesting that dyscalculia, or the mental disorder resulting in problems with counting, has anything to do with the workings

[19] M. Piazza, *Neurocognitive Start-Up Tools…*, *op.cit*., p. 276.

[20] *Ibid.*, pp. 275–276.

of OTS. Finally, experiments with infants and young children suggest that some features of OTS may be detrimental in numerical tasks. As pointed out above, OTS serves to track objects, 'taking notice' of their various characteristics (shape, size, colour, etc.): in some tasks, when such features (surface or contour length) were pitted against number, infants automatically attended to them and failed to recognize number. "It is extremely difficult to understand how a system that often interferes with numerical tasks might be relevant for learning yet more complex numerical representations."[21]

At the same time, it must be noted that the functioning of ANS alone is insufficient to account for the acquisition of counting skills. Even if Piazza is right, and the role of OTS is negligible in this process, ANS may only provide exact enough mechanism to learn the counting of relatively small numbers. As the Weberian ratio suggests, a 20-year-old person would be able to distinguish between 7 and 8, but not 8 and 9, so – if her counting abilities were based solely on ANS – she could use exact counting in the range of count from 1 to 8 only. This suggests that even if OTS plays no significant role in breaking the number 4 threshold on the top of ANS an additional mechanism is needed to explain the acquisition of natural number counting skills.

A hypothesis which addresses this problem is defended by Spelke. She observes that "children appear to overcome the limits of the core number system when they begin to use number words in natural language expressions and counting."[22] Children learn the first ten counting words by the age of 2, but initially use them without the intended meaning. At 3 they know that 'one' means one; at 4 they associate '2', '3' and '4' with the corresponding numerosities. Then, there is a kind of 'jump' – children learn quite quickly subsequent numbers. This, according to Spelke, requires two things: (a) to understand that every word in the counting list designates a set of individuals with a unique cardinal value; and (b) to grasp the idea that each cardinal value can

[21] *Ibid.*, p. 280.

[22] E.S. Spelke, *Natural Number*…, *op.cit.*, p. 304.

be constructed through progressive addition of 1.[23] How is this possible? "For most children, the language of number words and verbal counting appears to provide the critical system of symbols for combining the two core systems (i.e., ANS and OTS), and some evidence suggests that language may be necessary for this construction."[24]

Spelke's hypothesis that the development of counting skills is conditioned and mediated by the acquisition of language, is supported by the following pieces of evidence. First, both children and adults in remote cultures, whose languages have no words for numbers, recognize their equivalence only approximately when dealing with numbers larger than three. Second, an interesting line of evidence comes from the study of the mathematical abilities of deaf people. Deaf people living in numerate cultures but not exposed to the deaf community use a gestural system called homesign; they use their fingers to communicate numbers, but only with approximate accuracy. Similarly, they perform matching tasks with approximate accuracy.[25]

Spelke claims further that language continues to play an important role in mathematical cognition even after mathematical skills are mastered. For instance, educated adults who suffer language impairments have problems with exact, but not approximate numerical reasoning. Similarly, when doing exact (but not approximate!) tasks adults spend more time with numbers that are difficult to pronounce, even if they are presented in Arabic notation. But "if language merely scaffolded the acquisition of natural number concepts and abilities, and then was replaceable by other symbol systems, one would not expect adults to translate Arabic symbols into words for purposes of exact computation."[26] Finally, bilingual adults who are taught some new mathematical facts in one of their languages have difficulties in the smooth production of exact number facts in the other language.

23 *Ibid.*, p. 305.
24 *Ibid.*, p. 305.
25 *Ibid.*, pp. 306–307.
26 *Ibid.*, p. 307.

To be sure, there are serious doubts as to the extent of the role of language in the development of mathematical skills. Rochel German and Brian Butterworth believe that there is a "need to distinguish possession of the concept of numerosity itself (knowing that any set has a numerosity that can be determined by enumeration) from the possession of representations (in language) of particular numerosities."[27] Their case rests on the following observations. Firstly, they observe that the brain systems involved in numerical processing are 'some distance away' from the brain areas associated with the production of speech. Secondly, they are not convinced by the studies of Amazonian natives who have limited number vocabulary: their apparent lack of exact counting skills may result from the fact that they did not understand the tasks to be performed. In a latter study, Butterworth and colleagues compared arithmetic skills of three groups of children: English speaking from Melbourne, and two groups of indigenous Australians, speaking Warlpiri and Anindilyakwa. Warlpiri "has three generic types of number words: singular, dual plural, and greater than dual plural."[28] Anindilyakwa, in turn, "has four possible number categories: singular, dual, trial (which may in practice include four), and plural (more than three)."[29] The tests included: memory for a number of counters, cross-modal matching of discrete sounds and counters, nonverbal exact addition, and sharing play-dough disks that could be partitioned by the child. The results showed no 'language effects': all three groups performed similarly. This led Butterworth *et al.* to the conclusion that "the development of enumeration concepts does not depend on the possession of a number-word vocabulary. [Therefore] we are born with a capacity to represent exact numerosities, and (...) using words to name exact numerosities is useful

[27] R. Gelman, B. Butterworth, *Number and Language: How Are They Related?*, "TRENDS in Cognitive Sciences" 2005, vol. 9, no.1, p. 6.

[28] B. Butterworth, R. Reeve, F. Reynolds, D. Lloyd, *Numerical Thought with and without Words: Evidence from Indigenous Australian Children*, "Proceedings of the National Academy of Sciences" 2008, vol. 105, no. 35, (10.1073 PNAS 080604510), p. 13179.

[29] *Ibid.*, p. 13179.

but not necessary. When children learn to count, they are learning to map from their pre-existing concepts of exact numerosities onto the counting word sequence. Conceptual development drives the acquisition of counting words rather than the other way around."[30]

While Butterworth and his team's reservations seem well-founded, their conclusion seems to go too far. Firstly, the controversy does not concern the question of whether all of our mathematical abilities are inborn or else are enabled by language: we are speaking of relatively simple arithmetic tasks. It follows, secondly, that the thesis that language is crucial in the development of mathematical skills is not contested; rather, the extent of its role is at stake. Finally, Butterworth's team criticize the idea that all mathematical representations are conditioned on language representations or that – in the development of mathematical skills – language is *first* and *precedes* the emergence of mathematical mental concepts. However, this argument presupposes a certain view of language as based on an isolated system of mental representations. As I shall argue below, there exists a different approach to language, one regarding language as *not preceding* other forms of cultural representation, but rather *intertwined* with them. All in all, one should agree with Stanislas Dehaene who remarks:

> We all start out on the same rung, but we do not all climb to the same level. Progress on the conceptual scale of arithmetic depends on the mastery of a toolkit of mathematical inventions. The language of numerals is just one of the cultural tools that broaden the panoply of available cognitive strategies and allow us to resolve concrete problems.[31]

The claim that it is language and the other 'cultural tools' that shape our exact mathematical thinking is quite general. The question is,

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[30] *Ibid.*, p. 13182.

[31] S. Deheane, *The Number Sense*, 2nd edition, Oxford University Press, Oxford 2011, pp. 263–264.

how exactly the passage from elementary mathematical operations on small numbers – *via* language – to transfinite numbers, axiomatic set theory or non-Euclidean geometries is made. There are not many proposals as how to answer this question. The most notable one is George Lakoff and Rafael Núñez's idea of *embodied mathematics* presented in their study *Where Does Mathematics Come From?*[32]

Lakoff and Núñez's work belongs to the approach known as the embodied mind paradigm. The basic tenets of this stance are as follows. The very idea of embodiment boils down to the thesis that the human mind and human cognition are decisively shaped by the experiences of our bodies. This is a vague claim that only underlines the rejection of other paradigms, such as Cartesian mind-body dualism or computationalism (the rough idea that the human brain is hardware, and the mind is software implemented in the brain). However, other claims of the representatives of the embodied mind approach are more informative. They believe that the human mind is a powerful conceptual system shaped by our bodies' experiences during their interactions with the environment. The most basic mental concepts or schema, probably derived from the neural motor-control programs, express spatial relations (such as the Source – Path – Goal schema). Since such "image schemas are conceptual in nature, they can form complex composites. For example, the word 'into' has a meaning – the *Into schema* – that is the composite of an *In schema* and *To schema*."[33] Further, this mental machinery is capable of producing abstract concepts with the use of concrete ones through the use of metaphors. In the embodied paradigm metaphors are understood as the means for "understanding and experiencing one kind of thing in terms of another."[34] And so, importance is conceptualized in terms of size ("This is a big issue", "It's a small issue; we can ignore it"), dif-

[32] Cf. G. Lakoff, R. Núñez, *Where Mathematics Comes From?*, Basic Books, New York 2000.

[33] *Ibid.,* p. 39.

[34] *Ibid.*, p. 5.

ficulties are conceptualized as burdens ("He is overburdened", "I've got a light load this semester"), etc.[35]

> Each such conceptual metaphor has the same structure. Each is a unidirectional mapping from entities in one conceptual domain to corresponding entities in another conceptual domain. As such, conceptual metaphors are part of our system of thought. Their primary function is to allow us to reason about relatively abstract domains using the inferential structure of relatively concrete domains.[36]

Lakoff and Núñez claim further that it is the mechanism of conceptual metaphorization that enables the construction of complex and precise mathematical concepts. In the case of arithmetic, they postulate the existence of four basic or grounding metaphors: the Arithmetic as Object Collection (where the source domain concept of collections of objects of the same size is mapped to the concept of numbers, the size of the collection is mapped to the size of the number, the smallest collection is mapped to the concept of the unit, while putting collections together is mapped to the process of addition); the Arithmetic as Object Construction (where the source domain concept of objects consisting of ultimate parts of unit size is mapped to the concept of numbers or the act of object construction is mapped to the concept of arithmetic operations); the Measuring Stick metaphor (where physical segments are understood as numbers, the basic physical segment as one, and the length of the physical segment as the size of the number); and the Arithmetic as Motion Along a Path metaphor (where the act of moving along the path is understood as representing mathematical operations, point-locations on the path are understood as numbers, etc.).

Lakoff and Núñez claim that those four grounding metaphors give rise to the development of basic arithmetic. One begins with innate capacities to 'deal' with small numbers (up to 4). In addition, one

35 *Ibid.*, p. 41.
36 *Ibid.*, p. 42.

has primary experiences with object collections, object construction, physical segmentation and moving along a path. "In functioning in the world, each of those primary experiences is correlated with subitizing, innate arithmetic, and simple counting."[37] Those two domains are combined through the four metaphors in such a way that the primary experiences become sources of the metaphors and the domain of numbers is the target of the metaphors.

> The next step is the conflation among the primary experiences: object construction always involves object collection. Placing physical segments end to end is a form of object construction (…). From a neural perspective, [such conflations] involve co-activations of those brain areas that characterize each of the experiences. (…) As a consequence, an isomorphic structure emerges across the source domains (…), which is independent of numbers themselves and lends stability to arithmetic.[38]

The ability of subitizing – found in all normal human beings – leads to precise and stable results regarding small numbers; when extended with the four grounding metaphors, the precision and stability extends to all natural numbers. Finally, "the laws of arithmetic (commutativity, associativity and distributivity) emerge first as properties of the four source domains, then as properties of numbers via those metaphors, since the metaphors are inference-preserving conceptual mappings."[39]

Lakoff and Núñez use the same ideas to explain the emergence of more complex mathematical concepts, such as algebra, logic and set theory, real numbers, etc. There is no need to present here all those constructions, as they remain highly speculative. It is worthwhile, however, to have a look at one example: actual infinity.

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[37] *Ibid.*, p. 93.
[38] *Ibid.*, pp. 95–96.
[39] *Ibid.*, p. 96.

> We hypothesize – Lakoff and Núñez say – that all cases of actual infinity – infinite sets, points at infinity, limits of infinite series, infinite intersections, least upper bounds – are special cases of a single conceptual metaphor in which processes that go on indefinitely are conceptualized as having and end and an ultimate result. We call this metaphor the *Basic Metaphor of Infinity*. The target domain of the BMI is the domain of processes without end – that is, what linguists call imperfective processes. The effect of BMI is to add a metaphorical completion to the ongoing process so that it is seen as having a result – an infinite *thing*.[40]

In the case of BMI, the source domain are the iterative processes in which, although they do have a finite number of iterations, the number is indeterminate. The essential element of the metaphor is, however, to "add to the target domain the completion of the process and its resulting state."[41]

The theory developed by Lakoff and Núñez may be criticized in a number of ways. One can point out some mathematical errors in the book (e.g., the 'invention' of 'the first infinitesimal'). It may also be pointed out that Lakoff and Núñez seem to underestimate the role of ANS in arithmetic skills acquisition, while overplaying the role of OTS. One may even try to dismiss their approach claiming that these are 'just so stories', without any solid empirical grounds with the exception of some linguistic insights (e.g., the fact that we speak of zero using words which are connected to emptiness, lack, absence or destruction, and that number one has connotations with individuality, separateness, wholeness, integrity or beginning).[42] Further, Lakoff and Núñez's idea of schemes and concepts as constituting the 'deep structure of language' may be to some extent mistaken. However, all this misses the crucial point. Unlike other theorists, Lakoff and Núñez

[40] *Ibid.*, p. 158.
[41] *Ibid.*, p. 158.
[42] *Ibid.*, p. 75.

do offer an account of how to get from the limited innate mathematical capacities to full-blooded mathematics (even if they do not describe this process in the flawless way). Moreover, they show how it is possible to get to Cantor's paradise by starting with more *concrete* concepts, which reflects both phylogenetic and ontogenetic trajectories described in the literature.

All in all, Lakoff and Núñez's conception is a captivating idea and, in particular, their conception of *embodiment*: mathematical cognition is embodied in the sense that "it is grounded in simulations of sensorimotor processes through the use of neural resources that are also active in bodily perception and action." Embodied mathematics – in contrast to more modest approaches – does draw a line from the workings of OTS and ANS to our mathematical practices. It is a road that goes through our concrete bodily experiences as reflected in language; in a way, then, one can think of their theory as a concretization of Spelke's bold claim that the road to Cantor's paradise leads through language.

3. On How to Remain in Paradise or the Embedded Mathematics

The best way to assess the embodied approach to mathematics is to compare it with an alternative. Given that there seems to exist a strong link between the acquisition of language and the development of mathematical skills, it is advisable to see how the origins of mathematical concepts may be explained within different approaches to the evolution of language.

Let us begin with the Chomskyan idea of universal grammar. As well known, Chomsky claims that some basic grammatical structures are hard-wired in the brain: all languages must have a common structural underpinning. The question is whether the Chomskyan approach may be helpful in explaining where mathematics comes from, or, in other words, how to get from our inborn mathematical capacities (ANS and OTS) to our actual mathematical practices. It seems to me that the answer is a plain 'no'. There is no route from simple mathematical capacities to full-blooded mathematics through the assumption that we have some hardwired grammatical rules. Moreover, I believe that this argument can be generalized: any conception which posits some formal inborn structures (proto-language, universal grammar, proto-language) is incapable of paving the way from the embrained mathematics to Cantor's paradise. First, it seems that one cannot take advantage of such inborn formal capacities to break the 'number four threshold" and, in consequence, to understand how counting to natural numbers greater than 4 is possible. One may object here, claiming that if the proto-logic or proto-language has a built-in induction mechanism, it would explain the move from number 4 to larger numbers. However, the problem with breaking the 'number four threshold' has little to do with induction; the problem is rather how to grasp the idea that numbers larger than 4, represented approximately by the ANS system, are exact numbers in the same way numbers 1–4 are. Thus, it is rather a 'material' or 'substantive' issue than a formal one. Further, even if we assumed that some inborn universal formal structures may explain the acquisition of the arithmetic skills, the same cannot hold of geometry. It seems, therefore, that Chomsky-like theories of language cannot help us in accounting for the development of mathematical thinking.

Fortunately, there exists an alternative explanation of language evolution and acquisition. It is advocated by such scholars as Merlin Donald, Michael Tomasello, Michael Arbib and others.[43] The fundamental observation motivating this stance is that the genetic difference between the human species and other animals is not so large (appox. 1 – 1.2%). This constitutes an argument for the thesis that the

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[43] Cf. M. Tomasello, *Cultural Origins of Human Cognition*, Harvard University Press, Cambridge, Mass. 2001; M. Tomasello, *Why We Cooperate*, MIT Press, Cambridge, Mass. 2009; M. Arbib, *The Mirror System, Imitation, and the Evolution of Language*, [in:] *Imitation in Animals and Artifacts*, eds. Ch. Nehavin, K. Dautenhahn, MIT Press, Cambridge, Mass. 2002, pp. 229–280; M. Donald, *Imitation and Mimesis*, [in:] *Perspectives on Imitation*, vol. 2: *Imitation, Human Development, and Culture*, eds. S. Hurley, N. Chater, MIT Press, Cambridge, Mass. 2005, pp. 283–300.

biological adaptation enabling the flourishing of human culture must be relatively 'small'. In other words, the proponents of the described scenario claim that it is impossible to account for the development of various aspects of culture (language, morality, science, etc.) by recourse to a large number of biological adaptations. In particular, Michael Tomasello believes that:

> the 6 million years that separates human beings from other great apes is a very short time evolutionary, with modern humans and chimpanzees sharing something on the order of 99 percent of their genetic material – the same degree of relatedness as that of other sister genera such as lions and tigers, horses and zebras, and rats and mice. Our problem is thus one of time. The fact is, there simply has not been enough time for normal processes of biological evolution involving genetic variation and natural selection to have created, one by one, each of the cognitive skills necessary for modern humans to invent and maintain complex tool-use industries and technologies, complex forms of symbolic communication and representation, and complex social organizations and institutions.[44]

Tomasello further claims that

> there is only one possible solution to this puzzle. That is, there is only one known biological mechanism that could bring about these kinds of changes in behavior and cognition in so short a time (…). This biological mechanism is social or cultural transmission, which works on time scales many orders of magnitude faster than those of organic evolution.[45]

This, in turn, is made possible by three forms of learning: imitative, instructed and collaborative, which are conditioned by "a single spe-

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[44] M. Tomasello, *Cultural Origins…*, *op.cit.*, p. 2.
[45] *Ibid.*, p. 4.

cial form of cognition, namely, the ability of individual organisms to understand conspecifics as being *like themselves*, who have intentional and mental lives like their own."[46]

There are a number of facts that underline the distinctive human capacity to relate to conspecifics. The strongest such line of argumentation is connected to the differences between humans and other primates. In a number of experiments, Tomasello and his colleagues have demonstrated that the great apes' learning differs substantially from human ways of cultural transmission. In particular, apes learn by emulation (i.e., they grasp only the means-ends structure of an activity and do not copy the pattern of behaviour), while humans learn by imitation and instruction.

These observations suggest that the *ability to imitate* is one of the crucial adaptations in the evolutionary history of humankind. Tied with this is the capacity of '*mindreading*' or '*intention-reading*'. In addition, evolution has equipped us in a cluster of emotions (for Tomasello, guilt and shame are the basic emotions for cementing social bonds). These adaptations, taken together, are responsible for what Tomasello calls human *mutualism*.[47] We not only have the ability to take the perspectives of others; we can also take a perspective *with* others. To put it differently, humans not only understand what some other individuals do (their intentions), but also do things *together* with them (i.e., we are capable of *we-intentionality*). This, according to Tomasello, is the key for understanding the possibility of cumulative cultural evolution.

A similar evolutionary scenario was sketched by Merlin Donald in relation to the emergence of language. Donald claims that the sources of the human ability to use language are based on *mimetic skills*, which evolved some 2 million years ago. He distinguishes between mimicry, imitation and mimesis. Mimicry is a simple copying of some action, with no understanding of its goal. Imitation is more

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[46] *Ibid.*, p. 5.

[47] Cf. M. Tomasello, *Why We Cooperate*, *op.cit*.

abstract and flexible, as it takes into account the goal of the action. Finally, mimesis is defined as:

> the reduplication of an event for communicative purposes. Mimesis requires that the audience be taken into account. It also demands taking a third-person perspective on the actor's own behaviour. Some examples are children fantasy play, the iconic gestures used in a social context, and the simulation of a 'heroic' death during a theatrical performance.[48]

Mimetic skills are thus founded on the ability to imitate, which in turn is conditioned by the mimicry skills.

Donald identifies four main types of mimetic representation, which are key to the transmission and propagation of culture: (1) reenactive mime, characteristic of role-playing; (2) precise means--end imitation (as in learning how to fry an egg); (3) the systematic rehearsal and refinement of skill (as in learning how to drive a car); and (4) nonlinguistic gesture (as in learning how to dance). He claims further that these mimetic skills were the foundation for the emergence of language and all the other forms of culture. He stresses that his proposal differs from the traditional scenarios which condition the emergence of culture on the prior emergence of language (the *language first* theory). According to Donald, some forms of culture, based on the mimetic skills, must have *preceded* language and enabled its evolution (the *culture first* theory).

Donald's theory leads to profound consequences. First, he claims that the human mind is intimately linked to the society in which it flourishes. One can even say that it is *co-created* by the community. The communal practices are constitutive of the human mind, both in the phylogenetic and ontogenetic dimensions. Second, language is not an individual but a network-level phenomenon: its evolution resembles the evolution of an ecosystem rather than of a single organ-

48 M. Donald, *Imitation and Mimesis*, *op.cit*., p. 289.

ism. Third, it follows that "cognitive neuroscientists are unlikely to find an innate language acquisition device, and should redirect their investigations toward the powerful analogue processing systems out of which language can emerge in group interactions."[49]

Another analogous theory – but in the context of neuroscience – is advocated by Michael Arbib. He attempts to answer the question of what is the neuronal basis for a certain feature of language – *parity* – which manifests itself in our ability to recognize what our interlocutor wants to say.[50] He observes that Broca's area – traditionally considered as the region of the brain responsible for the production of speech – is one of the areas in which there is a complex system of mirror neurons. Thus, Broca's area is implicated in the production of various multimodal linguistic actions (utilizing the hands, face and voice). In connection to this, Arbib formulates the Neuron System Hypothesis: the parity condition is fulfilled due to the fact that Broca's area has been evolutionarily built upon a perception system responsible for the recognition and execution of manual actions. Arbib believes that the hypothesis is backed by both the arguments resulting from neuroimaging experiments (execution and perception of manual gestures activate neurons located within or in the proximity of Broca's area) and the anatomical facts (it is assumed that the F5 region in the macaques brains, where the mirror neurons were discovered, is an analogue of the Broadmann 44 area in human brain, which is a part of Broca's area).

Arbib also presents an evolutionary hypothesis pertaining to the probable development of mirror neurons. The first stage consisted in the emergence of mirror systems dedicated to the perception and execution of manual actions. In the second stage, those mirror neurons served as the basis for the development of the ability to imitate manual gestures: simple forms of such imitation are found in apes, more complex forms are exclusively human. The third stage was the emergence

[49] *Ibid.*, p. 294.
[50] Cf. M. Arbib, *The Mirror System…*, *op.cit.*

of pantomimic skills, and the fourth stage led beyond the simple reenactment of human behaviour (some gestures were recognized as standing for something else, e.g. waving one's hand was interpreted as imitating a flying bird). The fifth stage was the emergence of proto--signs, or conventional gestures which made pantomime more precise (e.g., they enabled to distinguish gestures representing birds and the process of flying). Finally, the sixth stage resulted in the development of proto-language, which emerged through the separation of conventionalized manual, mimic and vocal gestures from pantomime.[51]

Arbib claims further that if the above sketched evolutionary scenario is sound, the human brain is *language-ready*, but *does not 'have' language*. Thus, we are forced to reject Chomsky's theory of universal grammar which is a conception of a brain 'having language'. This conclusion is further reinforced by the following two observations: first, the ability to learn and use language is not confined to the spoken language, but embraces a combination of manual, vocal and mimetic skills.[52] Second, proto-language was based on proto-phrases: proto-words functioned as our sentences, not our words.

The major objection against this view of language – at least from the Chomskyan perspective – is that it may have problems dealing with the poverty of stimulus argument. Let me recall that Chomsky suggested his theory of universal grammar partly in response to the observation that children – in their development – are not exposed to a sufficient number of external stimuli in order to form – solely on this basis – a formal grammatical structure; thus, the argument runs, at least the most fundamental aspects of the structure must be innate.

The reply of the proponents of the imitation-based account of language rests on two theses. On the one hand, they claim that Chomsky failed to realize that spoken language is not an isolated system: it is a part of, and is based upon, a larger cluster of communication skills,

[51] See *ibid*.

[52] M. Arbib, *Mirror Neurons and Language*, [in:] *Handbook of the Neuroscience of Language*, eds. B. Stemmer, H.A. Withaker, Academic Press, London 2006, p. 238.

embracing, in particular, the language of gestures. Thus, the stimuli that need to be taken into account are not limited to vocal ones; to the contrary, communication vocal stimuli form just a small part of what a child is exposed to in the process of communication.

On the other hand, the mechanism of imitation is a powerful tool that – in certain circumstances – may lead to a kind of 'combinatorial explosion'. Giacomo Rizolatti claims that there are two types of mirror neuron resonance and, as a result, two types of imitation. The high-level resonance is used to mirror the *goal* of an action, while the low-level resonance copies the *way of acting*. According to Rizolatti, only the human brain takes advantage of both mechanisms, which enables imitation *sensu stricto*. In apes and (possibly) other animals only high-level resonance is used and this explains apes' learning by emulation only. The utilization of both high-level and low-level resonance plays a key role in the enhancement of the flexibility and stability of human social reaction, as it enables – through recombination – to use the same patterns of behaviour, learned by imitation, to realize different goals, or to realize one goal with different means.[53]

This theory may also explain the mechanism of 'novelty' in mathematics, or – if I may call it so – the Meno problem. The possibility of discoveries in mathematics are explicable by recourse to two facts. First, in order to develop new mathematical concepts, one utilizes the mechanism of metaphorization: a new metaphor may lead to the establishment of new mathematical facts. Second, the fact that we learn much of our mathematics through imitation suggests that through the combination of previously learned patterns of behaviour, new connections between mathematical means and ends may be established.

Interestingly, there are studies which suggest some close links between the conception of language and culture as originating with imitation and the 'embodied' paradigm as advocated by Lakoff and

[53] Cf. G. Rizolatti, *The Mirror Neuron System and Imitation*, [in:] *Perspectives on Imitation vol. 1: Mechanisms of Imitation and Imitation in Animals*, eds. S. Hurley, N. Chater, MIT Press, Cambridge, Mass. 2005, pp. 55–76.

Núñez. Lakoff and Gallese suggest[54] that "that the sensory-motor system has the right kind of structure to characterize both sensory-motor and more abstract concepts."[55] Their argument runs as follows. First, they observe that simple action concepts such as 'grasp' may well be characterized at the level of sensory-motor structure. Second, they claim that mirror neurons and other premotor and parietal neurons are 'multimodal', i.e. they respond to more than one modality. "Such multimodality meets the condition that an action-concept must fit both the performance and perception of the action."[56] Third, "multimodality is realized in the brain through *functional clusters*, that is, among others, parallel parietal-premotor networks. These functional clusters form *high-level units*—characterizing the discreteness, high-level structure, and internal relational structure required by concepts."[57] Fourth, they argue that in order to understand a concept one needs to imagine oneself or someone else doing what the concept refers to. They claim further that imagination is a mental simulation, "carried out by the same functional clusters used in acting and perceiving."[58] Therefore, any conceptualization of a concrete concept (e.g., grasping) *via* simulation requires the use of the same functional clusters used in the action and perception of grasping. Further,

> all actions, perceptions, and simulations make use of neural parameters and their values. For example, the action of *reaching* for an object makes use of the neural parameter of direction; the action of *grasping* an object makes use of the neural parameter of force. So do the concepts of *reaching* and *grasping*. Such neural parametrization is pervasive and imposes a *hierarchical structure* on the brain: the same parameter values that characterize the *internal*

[54] Cf. V. Gallese, G. Lakoff, *The Brain's Concepts: The Role of the Sensory-Motor System in Conceptual Knowledge*, "Cognitive Neuropsychology" 2005, vol. 22 (3–4), pp. 455–479.
[55] *Ibid.*, p. 455.
[56] *Ibid.*, p. 458.
[57] *Ibid.*, p. 458.
[58] *Ibid.*, p. 458.

> *structure* of actions and simulations of actions also characterize the *internal structure* of action concepts.[59]

The final step of the argument is to move from action and other concrete concepts to more abstract ones. In this context, Lakoff and Gallese recall Narayanan's conception of certain premotor structures called X-schemas, which fit the perceptual structure of the motor actions. Moreover, Narayanan observed that the X-schemas "have exactly the right structure to characterize the collection of concepts that linguists refer to as 'aspect' – concepts that characterize the structure of events and our reasoning about events."[60] In this way we found ourselves in a known territory – recall that the X-schemas are fundamental to Lakoff and Núñez's theory of embodied mathematics. In other words, it may be suggested that some version of the imitation-based paradigm in the study of language constitutes the neural and evolutionary complement of Lakoff and Núñez's conception of the embodied mind and embodied mathematics.

It should be added that there is a considerable body of evidence linking bodily processes and mathematical cognition. First, "studies on neural correlates of hand movements and action understanding of hand gestures point to an overlapping circuitry in the prefrontal and intraparietal regions with number processing."[61] Second, "studies conducted with repetitive Transcranial Magnetic Stimulation (rTMS) show excitability of hand muscles during different number processing tasks."[62] Third:

> behavioral studies on math learning provide evidence for better math learning when instruction is supported with hand gestures;

[59] *Ibid.*, p. 458.
[60] *Ibid.*, p. 470.
[61] F. Soylu, *Mathematical Cognition as Embodied Simulation*, [in:] *Proceedings of the 33rd Annual Conference of the Cognitive Science Society*, eds. L. Carlson, C. Hölscher, T. Shipley, Cognitive Science Society, Austin, TX 2011, p. 1213.
[62] *Ibid.*, p. 1213.

> higher problem solving performance when non-communicative hand gestures are allowed, compared to when hands are restricted; and non-communicative hand gestures during problem solving provide clues for misconceptions in conceptual understanding of arithmetic and algebra.[63]

The picture resulting from those observations is the following. The road from inborn mathematics to full-blooded mathematics leads through language. But language itself is not based solely on inborn grammatical structures. It is rather a part of a larger whole of communication and culture-generating skills which take advantage of the mechanism of imitation. If so, language is deeply rooted, or co-constituted by our social practices. In other words, in addition to being embodied (based on our bodily experiences), it is also *embedded* – it crucially depends on our communicating with others. If this is true of language, it is also true of our mathematical practices: they are embrained, embodied *and* embedded. Social interactions are not only *triggers* of mathematical concept-construction – they are *constitutive* of our mathematical knowledge.

It must also be stressed that the fact that our mathematics is embedded within social practices helps to explain why our mathematical practice is highly stable. Lakoff and Núñez claim that "the stability of embodied mathematics is a consequence of the fact that normal human beings all share the same relevant aspects of brain and body structure and the same relevant relations to their environment that enter into mathematics."[64] So, their claim is that, because our conceptual metaphors have the same source domain (our bodily experiences represented in the concrete concepts we use), or that the source domain is in a way *shared,* the resulting abstract mathematical concepts must also be shared, which explains why mathematics is so

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[63] *Ibid.*, p. 1213.

[64] G. Lakoff, R. Núñez, *Where Mathematics Comes From*, Basic Books, New York 2000, p. 352.

stable. There is a defect in this argument: the mechanism of metaphorization is such that it enables the development of numerous abstract concepts on the basis of the same concrete concept (as Lakoff and Núñez's examples clearly show). The question is, then, why people, when metaphorizing mathematics, pick the same (or similar) metaphors. The answer to this question is provided, I believe, by the theory of culture based on imitation. The fact that we learn cultural patterns of behaviour (and so also counting, theory-proving, etc.), contributes to the stability of our mathematical conceptual scheme: we use the same abstract metaphors, because we teach each other those metaphors, and we have an inborn capacity to learn in this imitative way and to propagate the learned patterns of behaviour among the members of our society.

There are also two philosophical corollaries of the theory sketched above. The first is that even if the road to Cantor's paradise is a bumpy one, it resembles more of a highway than a mountain route. Mathematics is not a stand-alone, separate body of knowledge. It is intimately intertwined with all that we call culture. It would be hopeless to try to distil the phylogenetic or ontogenetic path of the development of mathematical skills while not considering it an aspect of, or a 'fibre' in, the development of culture in general.

Secondly, the above presented theory – a highly speculative one, I admit – represents an interesting example of the use of the criterion of coherence within cognitive science. Neuroimaging data, experiments in developmental psychology, linguistic facts, evolutionary scenarios, etc – while taken in isolation – may usually be interpreted in a number of ways. However, when put together, they may strengthen or reinforce one another. This is, I submit, the case with the theory outlined here, in which the findings of neuroscience, linguistics and evolutionary theory contribute to a coherent picture of the origins of mathematical thinking.

4. Trimming Plato's Beard?

In the next two sections of the essay I will attempt to address two challenges that may be raised against the theory of embrained, embodied and embedded mathematics, or – as I like to call it – the 3E theory. These are: the problem of the necessity in mathematics, and the problem of the mathematicity of the universe.[65]

There is a dimension of mathematical and logical research that traditionally poses a challenge to any naturalistic accounts of the ontology of mathematical or logical objects. It is well captured in the following observation by Jan Łukasiewicz:

> Whenever I deal with the smallest logical problems, I always have the feeling that I am facing some powerful, incredibly coherent and enormously resistant structure. I cannot make any changes within it, I create nothing, but working hard I uncover new details, gaining eternal truths.[66]

Such views as the one expressed by Łukasiewicz above give rise to the development of mathematical Platonism (or realism), a view that "mathematics is the scientific study of objectively existing mathematical entities just as physics is the study of physical entities. The statements of mathematics are true or false depending on the properties of those entities, independent of our ability, or lack thereof, to determine which."[67]

[65] There is one more challenge to the 3E account which I do not discuss here: the 'miraculous abilities' of mathematical prodigies (including *idiot savants*) and mathematical geniuses. This is a complex issue, but I believe that the challenges it poses will ultimately be resolved through neuroscientific investigations. See especially, e.g., S. Deheane, *The Number Sense*, 2nd edition, Oxford University Press, Oxford 2011, pp. 129–159.

[66] J. Łukasiewicz, *W obronie logistyki. Myśl katolicka wobec logiki współczesnej*, "Studia Gnesnensia" 1937, vol. 15.

[67] P. Maddy, *Realism in Mathematics*, Oxford University Press, Oxford 1990, p. 21.

There are many forms of mathematical Platonism. In particular, one should distinguish between ontological Platonism (a view pertaining to the existence of mathematical objects) and semantic Platonism (an epistemological view that mathematical statements are true or false). Ontological Platonism is a stronger theory – it implies the semantic one, but the opposite implication does not hold. Thus, in what follows I shall concentrate on the stronger claim. Arguably, ontological Platonism in mathematics, although it comes in various incarnations, embraces the following three theses:

(The existence thesis) Mathematical objects (or structures) exist.
(The abstractness thesis) Mathematical objects are abstract, non-spatio-temporal entities.
(The independence thesis) Mathematical objects are independent of any rational or irrational activities of the human mind. In particular, mathematical objects are not our constructions.[68]

The key question is how the above formulated theses are justified. With no pretence to comprehensiveness, I posit that there are three kinds of arguments backing mathematical Platonism in its ontological version. The first one is the *semantic argument*, well captured by Balaguer, but formulated earlier by Frege:[69]

(1) Mathematical sentences are true.
(2) Mathematical sentences should be taken at their face value. In other words, there is no reason to believe that mathematical sentences, as they appear, are not what they really are, or that there is a deep structure of mathematical sentences which differs from their surface structure, of what they seem at their face.

[68] Cf. Ø. Linnebo, *Platonism in the Philosophy of Mathematics*, [in:] *The Stanford Encyclopedia of Philosophy* (Fall 2011 Edition), ed. E.N. Zalta, 2011, URL = <http://plato.stanford.edu/archives/fall2011/entries/platonism-mathematics/>.
[69] Cf. *ibid*.

(3) By Quine's criterion, we are ontologically committed to the existence of objects which are values of the variables in the sentences we consider true.
(4) We are ontologically committed to the existence of mathematical objects.
(5) Therefore, there are such things as mathematical objects, and our theories provide true descriptions of these things. In other words, mathematical Platonism is true.

The second argument defending mathematical Platonism is *the indispensability argument*, or the Quine/Putnam argument. Maddy summarizes it: "we are committed to the existence of mathematical objects because they are indispensable to our best theory of the world and we accept that theory."[70] And in Putnam's own words: "mathematics and physics are integrated in such a way that it is not possible to be a realist with respect to physical theory and a nominalist with respect to mathematical theory."[71] A reconstruction of this argument might appear as follows:

(1) By Quine's criterion, we are committed to the existence of objects which our best physical theories speak of.
(2) Our best physical theories are expressed with the use of the language of mathematics.
(3) Therefore, we are committed to the existence of mathematical objects.
(4) When one is a realist with respect to physical theories, one must also be a realist with respect to mathematics.
(5) Therefore, mathematical Platonism is true.

Finally, Gödel's *intuition-based argument* may be reconstructed in the following way:

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[70] P. Maddy, *op.cit.*, p. 30.
[71] H. Putnam, *What Is Mathematical Truth?* (1975), reprinted in H. Putnam, *Mathematics, Matter and Method*, Cambridge University Press, Cambridge 1979, p. 74.

(1) The most elementary axioms of set theory are obvious; as Gödel puts it, they "force themselves upon us as being true."[72]
(2) In order to explain (1), one needs to posit the existence of mathematical intuition, a faculty analogous to the sense of perception in the physical sciences.
(3) Not all mathematical objects are intuitable; but our belief in the 'unobservable mathematical facts' is justified by the consequences they bring in the sphere controllable by intuition and through their connections to already established mathematical truths. As Gödels says, "even disregarding the [intuitiveness] of some new axiom, and even in case it has no [intuitiveness] at all, a probable decision about its truth is possible also in another way, namely, inductively by studying its 'success' (…). There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems (...) that, no matter whether or not they are [intuitive], they would have to be accepted at least in the same sense as any well-established physical theory."[73]

Let us consider now whether the 3E account of mathematics defended in this essay has any bearing on the arguments favouring mathematical Platonism. Lakoff and Núñez believe that the conception of the embodied mathematics puts mathematical Platonism to eternal rest. For them, mathematical Platonism is "the romance of mathematics", a "story that many people *want* to be true"; a story that mathematical objects are real, and mathematical truth is universal, absolute, and certain. They succinctly reject this view:

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[72] K. Gödel, *What Is Cantor's Continuum Problem?* (1947), reprinted in *Philosophy of Mathematics*, eds. P. Benacerraf, H. Putnam, Cambridge University Press, Cambridge 1983, p. 484.
[73] *Ibid.*, p. 477.

> The only access that human beings have to any mathematics at all, either transcendent or otherwise, is through concepts in our minds that are shaped by our bodies and brains and realized physically in our neural systems. For human beings – or any other embodied beings – mathematics *is* embodied mathematics. The only mathematics we can know is mathematics that our bodies and brains allow us to know. For this reason, the *theory of embodied mathematics* (…) is anything but innocuous. As a theory of the only mathematics we know or can know, it is a theory of what mathematics *is* – what it really is![74]

As I read them, Lakoff and Núñez underscore two things. First, they put forward an epistemological claim that we have no cognitive access to independent abstract objects, since the only way of practicing mathematics is through the concepts "shaped by our bodies and brains." The problem of the cognition of abstract objects has been a subject of controversy since the beginnings of philosophy. Painting with a broad brush, one may claim that two solutions have been defended in this context, both already present in Plato: that there exists a rational intuition enabling us to contemplate abstract objects or that our access to the abstract sphere is discursive, mediated by language. Lakoff and Núñez seem to consider only the first option, and dismiss it on the basis of recent findings in neuroscience.

Second, they seem to embrace a version of Quine's criterion: we are committed to the existence of those things only, of which our best scientific theories speak. They add that the best – or rather: the only – theory of mathematical cognition we have is the theory of embodied mathematics, and since it does not speak of independent abstract objects, we have no grounds for postulating their existence. The problem is that Quine's criterion – applied to other theories, not necessarily accounting for the nature of mathematics, e.g. our best physical

[74] G. Lakoff, R. Núñez, *Where Mathematics Comes From*, *op.cit.*, p. 346.

theories – brings a different outcome: that we are indeed committed to the existence of abstract mathematical objects.

To put it differently: it seems that Lakoff and Núñez's stance does not invalidate any of the three arguments in favour of mathematical Platonism described above. In order to defeat the semantic argument, Lakoff and Núñez would either have to show that mathematical statements cannot be ascribed truth or falsehood (which they do not do); or to reject the idea that mathematical statements have no 'deep structure', that they are what they seem at their face (which they do not do as well); or to reject Quine's criterion of ontological commitment (which they *embrace* in their argument). Given the main idea behind the embodied approach, i.e. that mathematical concepts are the outcomes of metaphorization, their 'best shot' would probably be the rejection of the second claim (that there is no deep structure to mathematical statements). However, the distinction between the surface and deep structures of statements hangs together with some conception of the *form* of those expressions. It is the *form* of an expression that constitutes its deep structure. Thus, were the distinction to survive at all, Lakoff and Núñez would need to introduce a very peculiar conception of the form of statements which lies *beneath* mathematical structures, one very difficult to imagine given that mathematics *is* the science of structures. Also, Lakoff and Núñez do not address the indispensability argument. To do so, they would need either to reject Quine's criterion; or the thesis that mathematical physics is our best theory of the world; or the realist stance towards physical theories. Again, this is difficult as they embrace Quine's criterion themselves and seem to be realists with respect to biological theories. Finally, the intuition-based argument seems the easiest to attack from the point of view of the embodied paradigm. As we have seen above, human 'intuitive' mathematical capacities are substantially limited. However, Gödel – the proponent of the intuition-based argument – does not claim that our intuition is a faculty that gives us access to the entire world of mathematical structures. His thesis is that intuition is the source of certainty in relation to relatively simple mathematical structures and relations; more complicated mathematical statements are evaluated as true because they are justified by commonly accepted mathematical methods and have consequences controllable at the intuitive level. Of course, Lakoff and Núñez may claim that the intuition Gödel speaks of is not an intuition of *abstract* objects; it is rather the capacity to use abstract mathematical concepts, which are ultimately shaped by the experiences of our bodies. But this criticism can be softened by a modification of Gödel's argument: instead of speaking of intuition, one can simply speak of mathematical experience, even conceived of in terms of Lakoff and Núñez's theory. The crux of Gödel's argument, or so I argue, lies somewhere else: mathematical Platonism is true, because "there exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems that, no matter whether or not they are [a subject of direct experience], they would have to be accepted at least in the same sense as any well-established physical theory." Gödel points to something important here: the full power of our abstract conceptions, which lies beyond any intuitive or 'direct' experience, is clearly visible in the consequences they produce within the sphere controllable by experience, as well as in the coherence they bring to entire areas of mathematics and the heuristic role they play in solving mathematical problems. It is reasonable, therefore, to assume that those highly abstract concepts describe some independently existing structures rather than claim that they are just 'metaphorizations' of more concrete concepts. The mathematics we can somehow experience directly is only the tip of the iceberg: and when Lakoff and Núñez believe that the rest of the iceberg is only an illusion, Gödel seems to claim that it is a rock-hard, even if abstract, reality.

All this is not to say that the three arguments supporting mathematical Platonism are irrefutable or incontestable: the heated debates in the philosophy of mathematics during the last century are the evidence to the contrary. However, Lakoff and Núñez failed to provide a persuasive case against mathematical Platonism. It does not change the fact that there exists a *tension* between the embodied paradigm and mathematical Platonism: the former stresses that mathematics is *our construction*, when the latter underscores that mathematics is *independent of us*.

5. The Mathematicity of the Universe

Michael Heller introduces the concept of the mathematicity of the universe in the following words:

> In the investigation of the physical world one method has proved particularly efficient: the method of mathematical modeling coupled with experimentation (to simplify, in what follows I shall speak of the mathematical method). The advances in physics, since it has adopted the mathematical method, have been so enormous that they can hardly be compared to the progress in any other area of human cognitive activity. This incontestable fact helps to make my hypothesis more precise: the world should be ascribed a feature thanks to which it can be efficiently investigated with the use of the mathematical method. Thus the world has a rationality of a certain kind – mathematical one. It is in this sense that I shall speak of the mathematicity of the universe.[75]

According to Heller, to say that the world is mathematical is equivalent to the claim that it possesses a feature which makes the mathematical method efficient. In the quoted passage, Heller hints at one of the aspects in which the mathematicity of the world should be understood: the Efficiency Thesis. It says that the mathematicity of the universe is evident once one considers the enormous success of the mathematical method during the last 300 years. The success cannot be a pure coincidence, as the efficiency of mathematics in uncovering

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[75] M. Heller, *Czy świat jest matematyczny?*, [in:] *idem*, *Filozofia i wszechświat*, Universitas, Kraków 2006, p. 48.

the laws of nature seems 'unreasonable'.[76] The argument pertaining to the 'unreasonable effectiveness of mathematics' is not trivial. As Eugene Wigner observes:

> It is true, of course, that physics chooses certain mathematical concepts for the formulation of the laws of nature, and surely only a fraction of all mathematical concepts is used in physics. It is true also that the concepts which were chosen were not selected arbitrarily from a listing of mathematical terms but were developed, in many if not most cases, independently by the physicists and recognized then as having been conceived before by the mathematicians. It is not true, however, as is so often stated, that this had to happen because mathematics uses the simplest possible concepts and these were bound to occur in any formalism. [Moreover], it is important to point out that the mathematical formulation of the physicist's often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena. This shows that the mathematical language has more to commend it than being the only language which we can speak; it shows that it is, in a very real sense, the correct language.[77]

There are some phenomena connected to the use of the mathematical method that lead to the conclusion that it is some feature of the world that must be responsible for the method's successes. It is often the case that mathematical equations describing some aspects of the universe 'know more' than their creators. The standard story in this context is that of Einstein's cosmological constant. When Einstein formulated his cosmological equations on the basis of the newly discovered general relativity theory, he realized that they implied a dynamic, expanding universe. In order to 'stop' the expansion, he in-

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[76] Cf. E. Winger, *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*, "Communications on Pure and Applied Mathematics" 1960, vol. 13(1), pp. 1–14.
[77] *Ibid.*, p. 7.

troduced the cosmological constant. It quickly proved, however, that Einstein was 'wrong' and his equations were 'right': the expansion of the universe is a fact.

Another instructive example is given by Wigner. When Heisenberg formulated his quantum mechanics based on matrix calculus, the theory was applicable only to a few idealized problems. Applied to the first real problem, of the hydrogen atom, it also proved successful:

> This was (...) still understandable because Heisenberg's rules of calculation were abstracted from problems which included the old theory of the hydrogen atom. The miracle occurred only when matrix mechanics, or a mathematically equivalent theory, was applied to problems for which Heisenberg's calculating rules were meaningless. Heisenberg's rules presupposed that the classical equations of motion had solutions with certain periodicity properties; and the equations of motion of the two electrons of the helium atom, or of the even greater number of electrons of heavier atoms, simply do not have these properties, so that Heisenberg's rules cannot be applied to these cases. Nevertheless, the calculation of the lowest energy level of helium (...) agrees with the experimental data within the accuracy of the observations, which is one part in ten million. Surely in this case we 'got something out' of the equations that we did not put in.[78]

The second aspect of the mathematicity of the universe may be called *the Miracle Thesis*. It is possible to imagine worlds which are *mathematical* in a certain sense, yet non-idealizable. Michael Heller considers a hierarchy of such worlds. 'The most non-mathematical' is a world in which no mathematical and logical principles are observed (including any stochastic or probabilistic laws). Next, he suggests to consider a simplified model of the world: let us assume that the world in question may be in one of only two states, represented by '0' and '1'. Now:

[78] *Ibid.*, p. 10.

> The history of this world is thus a sequence of ‘0’s and ‘1’s. Assume further that the world had a beginning, what may be represented by a dot at the beginning of the sequence. In this way, we get, e.g., a sequence:
>
> .011000101011...
>
> The task of a physicist is to construct a theory which would enable us to predict the future states of the world. Such a theory would amount to the ‘encapsulation’ of the sequence of ‘0’s and ‘1’s in a formula (which is shorter than the sequence it encapsulates). Such a formula may be found only if the sequence of ‘0’s and ‘1’s is algorithmically compressible. But this leads to a problem. Such a sequence may be interpreted as a decimal expansion of a number in [0,1] and – as well known – the set of algorithmically compressible numbers belonging to [0,1] is of measure 0 (…). Thus (…) there is zero-measure chance that a sequences of ‘0’s and ‘1’, representing our world, belongs to the set of algorithmically compressible sequences and so the physicist, who investigates such a world, may have no rational expectation to discover the theory she is looking for.[79]

This observation underscores ‘the other side’ of the mathematicity thesis: not only is universe mathematical (and hence penetrable by *some* mathematical method), but it is also mathematical in a non-malicious way (and hence penetrable by *our* mathematical methods).

In connection to the problem of the mathematicity of the world, Lakoff and Núñez claim:

> No one observes laws of the universe as such; what are observed empirically are regularities in the universe (…); laws are mathematical statements made up by human beings to attempt to characterize those regularities experienced in the physical universe. (…) What [the physicists] do in formulating ‘laws’ is fit their hu-

[79] M. Heller, *Czy świat jest matematyczny…*, *op.cit*., pp. 51–52.

> man conceptualization of the physical regularities to their prior human conceptualization of some form of mathematics. All the 'fitting' between mathematics and physical regularities of the physical world is done within the minds of physicists who comprehend both. The mathematics is in the mind of the mathematically trained observer, not in the regularities of the physical universe.[80]

This, again, is an example of bad philosophy. Núñez and Lakoff fail to realize the far-reaching consequences of the Efficiency Thesis. What they leave unaccounted for are, at least, the fact that the mathematical method helped us to conquer the micro-scale phenomena; that equations often 'know more' than their creators; that mathematical models are often the basis for formulating qualitatively *new* predictions, and so serve as a powerful heuristic tools. It seems that behind Lakoff and Núñez's observations lies a very simplistic or naïve view of science, which rests on the *observations* of the regularities of real-world phenomena and their generalizations into the mathematically expressible laws of physics. What follows, within Lakoff and Núñez's framework one cannot even formulate the Miracle Thesis.

The interesting fact is that a conception of mathematics which draws on Lakoff and Núñez's work may shed some light on the Efficiency Thesis. The argument is quite general. Both our inborn mathematical capacities, as well as our conceptual apparatus have been shaped – in the evolutionary process – by our interaction with the environment. Now, given that our environment is mathematical (in Heller's sense of the word), it helps us to understand why our mathematical concept are efficient in uncovering the laws of the universe. Of course, such an argument cannot explain fully the efficiency of mathematics in quantum physics, or the fact that physical equations sometimes 'know more' than their creators. However, it may serve to dismiss the idea that "all the 'fitting' between mathematics and physical regularities of the physical world is done within the minds of physicists

[80] Lakoff, Núñez, *op.cit.*, p. 344.

who comprehend both. The mathematics is in the mind of the mathematically trained observer, not in the regularities of the physical universe." The mind is mathematical because it is a part of the mathematical universe.

6. The Triple E and the Capital M

We face a twofold problem: how to account for the necessity of mathematics on the one hand, and how to explain its efficiency in uncovering the structure of reality on the other. It seems that the 3E theory of mathematics described above offers no acceptable answers to these questions. I posit, therefore, that it should be amended, and the best way to amend it is to take advantage of Karl Popper's conception of three worlds.

> In this pluralistic philosophy the world consists of at least three ontologically distinct sub-worlds; or, as I shall say, there are three worlds: the first is the physical world or the world of physical states; the second is the mental world or the world of mental states; and the third is the world of intelligibles, or *ideas in the objective sense*; it is the world of possible objects of thought: the world of theories in themselves, and their logical relations; of arguments in themselves; and of problem situations in themselves.[81]

One can add: of mathematical relations, functions, sets, etc. I will leave aside the numerous objections raised against Popperian theory but instead I would like to concentrate on a number of its features which I believe to be essential to the discussion pertaining to the view of mathematics that is proposed here.

Firstly, world 3 really exists. One should note, however, that Popper defines existence in a special, albeit not unprecedented way: exist-

[81] K. Popper, *Objective Knowledge*, Oxford University Press, Oxford 1972.

ing objects have the capacity to influence one another.[82] Popper says: "The theory itself, the abstract thing itself, I regard as real because we can interact with it – we can produce a theory – and because the theory can interact with us. This is really sufficient for regarding it as real."[83] He develops this thought as follows: "[One need] only think of the impact of electrical power transmission or atomic theory on our inorganic and organic environment, or of the impact of economic theories on the decision whether to build a boat or an aeroplane",[84] in order to reject the fictitious character of the world 3 objects.

Secondly, the world 3 is autonomous. "By this – explains Popper – I mean the fact that once we have started to produce something – a house, say – we are not free to continue as we like if we do not wish to be killed by the roof falling in."[85] The autonomy of the world 3 is connected to its objectivity. Both the autonomy and the objectivity are indicated by the fact that "certain problems and relations are unintended consequences of our inventions, and that these problems and relations may therefore be said to be discovered by us, rather than invented: we do not invent prime numbers."[86] In other words, Popper indicates that any 'discovery' within the world 3 may lead to some objective consequences which are independent of our will. For example, one of the consequences of the development of Frege's logical calculus in *Begriffschrift* was the possibility of constructing the Russell paradox. Frege was unaware of this possibility; on the other hand, Russell did not invent it, he only 'discovered' it. Besides, Popper claims that the distinction between 'invention' and 'discovery' is – in most contexts – unimportant, "for every discovery is like an invention in that it contains an element of creative imagination."[87] Be that as it may, the objectivity of the world 3 is, in Popper's view, indisputable.

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[82] *Ibid.*, p. 200.
[83] K. Popper, *Knowledge and the Body-Mind Problem*, Routledge, London 1996, p. 47.
[84] K. Popper, *Objective Knowledge…*, *op.cit.*, p. 159.
[85] K. Popper, *Knowledge and the Body-Mind Problem…*, *op.cit.*, p. 47–48.
[86] *Ibid.*, p. 48.
[87] *Ibid.*, p. 48.

Thirdly, Popper provides us with an evolutionary explanation for the emergence of the world 3. He believes – *contra* Plato – that the entities of the world 3 are not 'superhuman, divine and eternal'; they are the products of the long process of human evolution. The world 3 is a human product, in the same way as nests and dams are animal products. It is the expression of our adaptation, whose roots lie in our biology. The essential element of the evolutionary theory of the world 3 is its emergent character. Popper utilizes the classical understanding of emergence: "in the course of evolution new things and events occur, with unexpected and indeed unpredictable properties; things and events that are new, more or less in the sense in which a great work of art may be described as new."[88] Emergence leads then to new properties, which are irreducible to the properties of the underlying system.

One would be mistaken, however, to claim that the Popperian thesis that the emergent properties are irreducible is indefeasible. Although Popper stresses the implausibility of a future reductionist explanation of the emergence of life, language or mind, he confesses:

> I want to make clear that as a rationalist I wish and hope to understand the world and that I wish and hope for a reduction. At the same time, I think it quite likely that there may be no reduction possible; it is conceivable that life is an emergent property of physical bodies.[89]

This declaration is highly characteristic of Popper's ontological theory. Nowhere does he claim that the three-worlds ontology should be taken literally. He suggests only that – compared to other ontological conceptions – it is a better, more useful tool of philosophical argumentation. The world 3 is a 'useful convention'. "I would say – stresses Popper in *Knowledge and the Body-Mind Problem* – that really the name 'world 3' is just a way of putting things, and the thing

[88] K. Popper, J.C. Eccles, *The Self and Its Brain*, Routlege, London 1984, p. 22.
[89] K. Popper, *Objective Knowledge…*, *op.cit*., p. 292.

is not to be taken too seriously. We can speak about it as a world, we can speak about it as just a certain region."[90] In another essay he adds:

> Whatever one may think about the status of these three worlds – I have in mind such questions as whether they 'really exist' or not, and whether world 3 may be in some sense 'reduced' to world 2, and perhaps world 2 to world 1– it seems of the utmost importance first of all to distinguish them as sharply and clearly as possible. (If our distinctions are too sharp, this may be brought out by subsequent criticism).[91]

The world 3 "is a metaphor: we could, if we wish to, distinguish more than three worlds."[92] Or: "whether or not you distinguish further regions or worlds, is really only a matter of convenience."[93] In this way, Popper tries to say that the conception of the world 3 is *a step in the right direction*. Ultimately, it may transpire that it is better to speak of one, two, or forty-seven worlds. It is crucial, however, to realize that *vis a vis* the existing ontological theories, the idea of the world 3 constitutes progress. Put otherwise: the 'division' of the reality into three worlds is *a heuristic device*. It helps us to identify *real problems* and to appreciate the role of our *theories*.

Moreover, I believe that Popper's conception should be modified in an important way. Let us observe that for Popper the emergence of the world 3 is genetically connected to the emergence of language. Without language, there would exist no such world. Thus – given the imitation-based theory of language I sketched above – one can argue that the world 3 is founded not only on what goes on in our minds, but is co-constituted by our social interactions, as it is through the social interactions that some patterns of behaviour become parts of our common cultural heritage (i.e. of the world 3). Thus, the main

[90] K. Popper, *Knowledge and the Body-Mind Problem…*, *op.cit.*, p. 17.
[91] K. Popper, *Unended Quest*, Routlege, London 2002, p. 211.
[92] K. Popper, *Knowledge and the Body-Mind Problem…*, *op.cit.*, p. 25.
[93] *Ibid.*, p. 18.

modification of Popper's theory I propose is the rejection of the view that the world 3 emerges from the world 2 alone. I believe that a better hypothesis reads that the world 3 supervenes both on mental states (belonging to the world 2) and social interactions (which belong to the world 1).

Let us repeat the characteristic features of the world 3: it exists in the sense of exercising influence on other objects; it is autonomous, yet we created it; and it consists of abstract objects. On this view, the tension between the fact that mathematics is independent of us, while being our creation, diminishes. In other words, I believe that the Popperian ontology provides an answer to the problem of the necessity of mathematics.

Still, on the presented view mathematics is only a part, even if a designated one, of the world 3. It means that mathematical objects are not *sui generis*; in this way, the proposed conception of mathematics escapes the argument from queerness, often raised against various kinds of Platonism. According to the argument, were there Platonic objects (e.g., values, norms, ideas) they would be queer entities, dissimilar to anything else we know. Now, when the world 3 embraces theories, values, mathematical objects, logical relations, rules, etc, they no longer are as queer as mathematical objects or values considered as *sui generis* entities.

More importantly, the claim that mathematics forms a part of the world 3 is coherent with the imitation-based view of language and culture, as so with the 3E conception of mathematics. Our culture-creating abilities, based – *inter alia* – on our imitative skills, are instrumental not only in producing mathematical knowledge, but all kinds of knowledge. Therefore, it is only justified that the products of those activities (mathematical theories, ethical directives, physical theories, etc.) fall within the same ontological category: the world 3.

The bigger problem is connected to the mathematicity of the universe. Here also, however, Popper's ontology constitutes an interesting philosophical stance. According to Popper, science, in its development, asymptotically approaches truth. Truth is, therefore, a regulatory idea of science. We will never end our scientific quest claiming that our theories are true. We would have no subjective certainty regarding their truthfulness even if they were objectively true. This means, however, that Popper is – as Stanisław Wszołek puts it – 'a transcendental essentialist':[94] he believes that the universe has a structure and our attempts at deciphering it continuously bring better results or capture some aspects of the structure of the universe. Those attempts are guided by the mathematical method, and hence the transcendental essence of the universe must be mathematical. This explains the 'unreasonable effectiveness of mathematics in the natural sciences'.

Michael Heller frames his reply in a slightly different way:

> It is obviously true that genetically our mathematics comes from the world: we abstract some of its features. However, one needs to carefully distinguish between *our mathematics* and *mathematics as such*. Our mathematics (which I also deem 'mathematics with a small m') has been developed by humans in a long evolutionary process: it is expressed in a symbolic language we invented; its results are collected in our scientific journals, books, or computer memory. But our mathematics is only a reflection of certain relations or structures, which governed the movement of atoms and stars long before biological evolution began. I deem those relations or structures mathematics as such (or 'Mathematics with a capital M'); it is what we think of when we ask, why nature is mathematical. The answer to this question, which posits that the nature is mathematical because mathematics has been abstracted from nature, turns out helpless, or even naïve, when one introduces the distinction between our mathematics and mathematics as such.[95]

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[94] Cf. S. Wszołek, *Esencjalizm transcendentalny K.R. Poppera*, "Zagadnienia Filozoficzne w Nauce" 2002, vol. XXXI, pp. 120–132.

[95] M. Heller, *Co to znaczy, że przyroda jest matematyczna?*, [in:] *Matematyczność przyrody*, eds. M. Heller, J. Życiński, Petrus, Kraków 2010, p. 16.

Heller's claim is, then, that mathematics (with the small m) is so efficient in the process of investigating the universe, because it is a reflection of Mathematics (with the capital M), a feature of the universe. Moreover, the fact that mathematics does *resonate* with Mathematics is not as unreasonable as may seem at first: ultimately, our mathematical theories have been developed in constant interactions with the Mathematical universe.

Two things should be stressed here. Firstly, the above remarks are not intended to say that our mathematics is in a way *identical* to some part of Mathematics. The idea is only that the Popperian world 3, with our mathematical theories, is somehow capable of grasping aspects of the Mathematical universe. Secondly, even if the presented conception constitutes a philosophical reply to the Efficiency Thesis, it becomes powerless in the face of the Miracle Thesis: the universe could have been Mathematical in such a way, that the mathematics required to capture its structure would be too difficult for any carbon-based creatures to develop. This fact calls for a transcendental, if not theological reflection.

The picture resulting from the above considerations may be deemed 'The Triple E and a Capital M'. I posit the distinction between:

(1) The 'embrained' mathematics, i.e. a set of inborn basic mathematical skills (like ANS and OTS).
(2) The embodied mathematics, i.e. a conceptual system – based on our linguistic metaphorical schemes – which enables to expand human inborn mathematical capabilities.
(3) The embedded mathematics, i.e. a system of socially shared patterns of behaviour (concerning performance of various mathematical operations), propagated through imitation and enabling the stability of our mathematical knowledge.
(4) The transcendent Mathematics (with the capital M), i.e. a feature of the universe that makes possible the investigation of nature with the use of the mathematical method, as well as the evolutionary development of the mathematical brain.

This 'hierarchy' fits well the modified Popperian ontological stance: the three levels of mathematics ('embrained', embodied and embedded) give rise to our mathematical theories (with a small m), which captures some aspects of Mathematics (with the capital M). It must be stressed again that the distinction between the 'embrained', embodied and embedded mathematics (and especially between the latter two) is only analytical: we create mathematics thanks to our inborn abilities, coupled with the creative force of metaphorization enabled by language and other culture-creating skills, and sustained over time by our tendency to imitate.

Conclusion: Is the Empirical Subject Back?

To repeat: the view of mathematics defended here is that of the 3E theory: we began our mathematical journey in phylogeny, and each of us begins it in ontogeny, with some inborn mathematical skills, which are later enhanced thanks to the development of new concepts *via* metaphorization (or something akin to it) and sustained through social interactions. Our road to Cantor's paradise leads through our bodily experiences, but also through our social institutions: mathematics is, in a nontrivial sense, a joint enterprise – not only do many people contribute to the development of mathematics, but the 'small m' mathematics is co-constituted by our shared practices. Thus, one may say, there is no mathematics of the empirical subject, but rather of empirical subjects. Still, the necessity inherent in mathematics, as well as its remarkable efficiency in uncovering the structure of reality, leave us face to face with a mystery; a mystery that calls for ontological explanation and, in some cases, perhaps even for a theological one. While further advances in neuroscience and related disciplines will inevitably lead to our better understanding of what mathematical skills consist in, and – in effect – what mathematics is, some problems, such as the one summarized by the Miracle Thesis, will remain subject to insightful philosophical reflection.